Beyond the Flat World: PDE's on Manifolds

A Geodesics Approach

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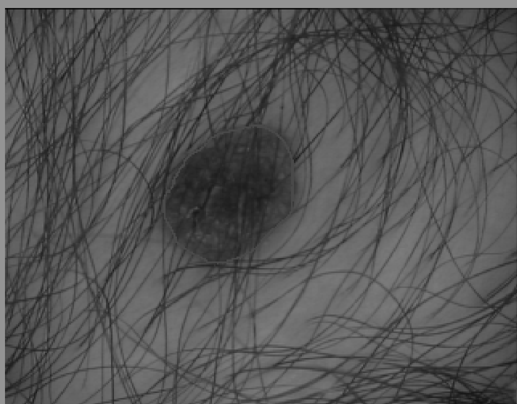
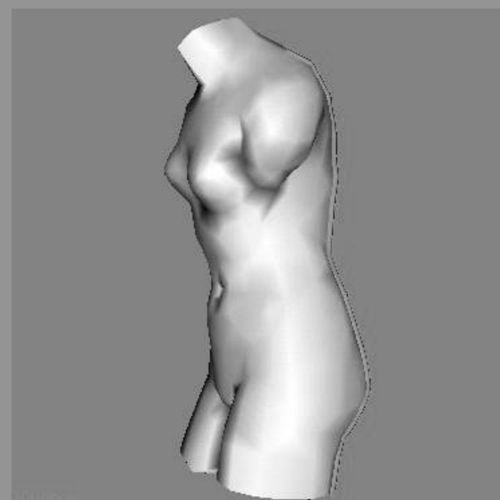
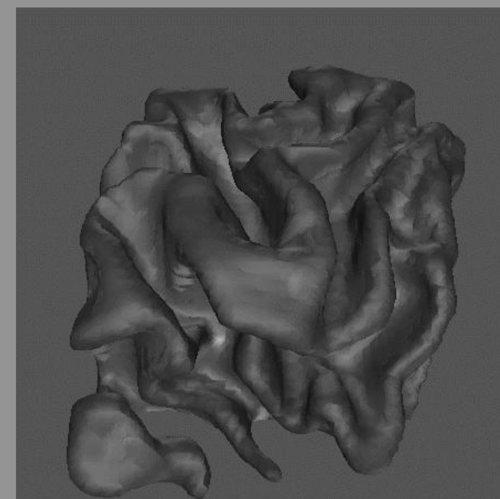
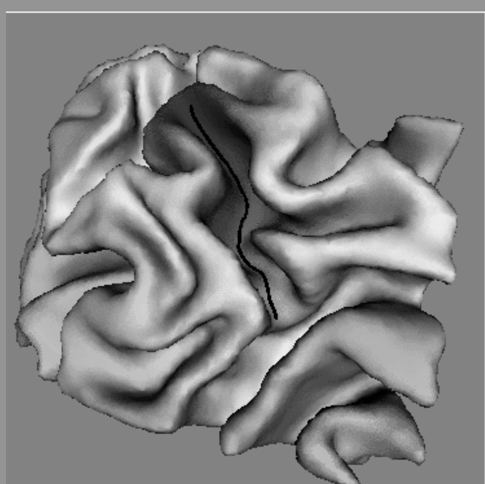
Overview

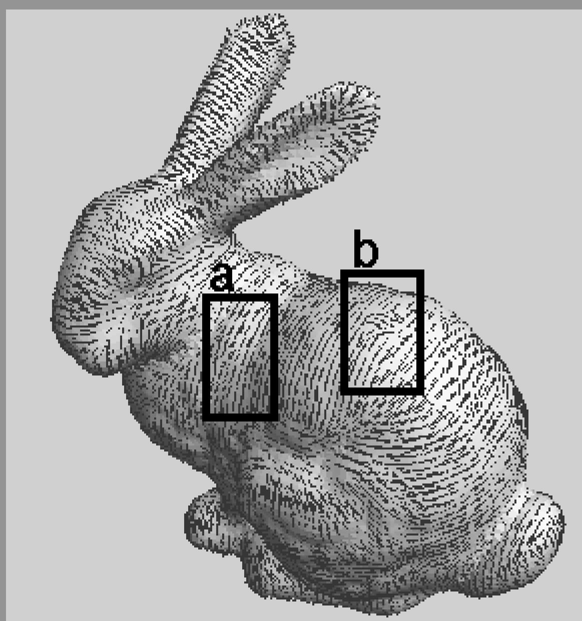
- Motivation
- Background on fast/accurate geodesic computations
- Distance functions and geodesics on implicit hyper-surfaces
- Unorganized points
- Generalized geodesics
- The future and concluding remarks

Motivation: A Few Examples

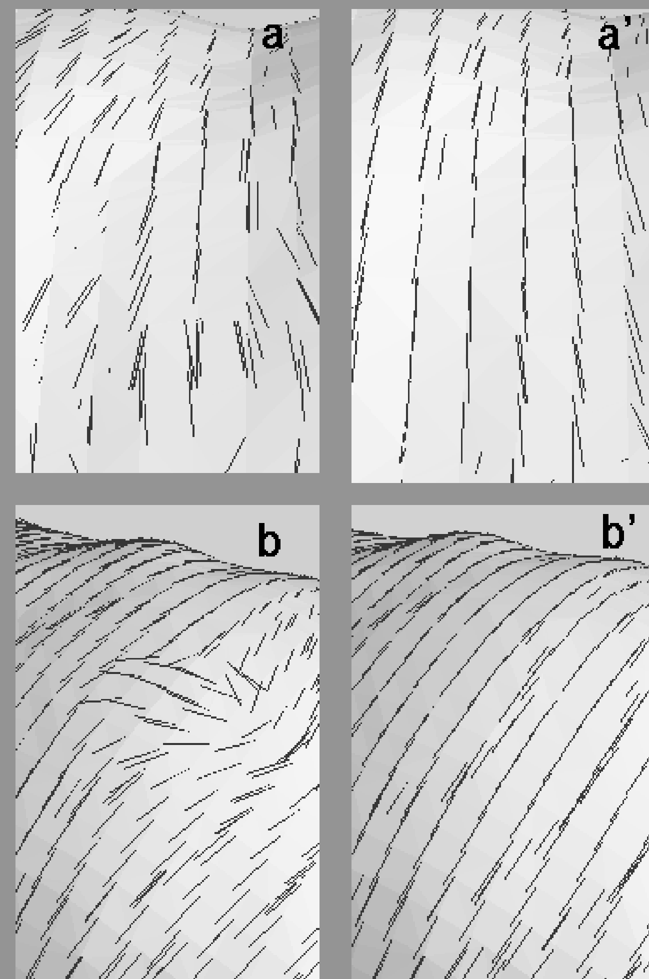





Show me!!!

Motivation: A Few Examples (cont.)



noisy



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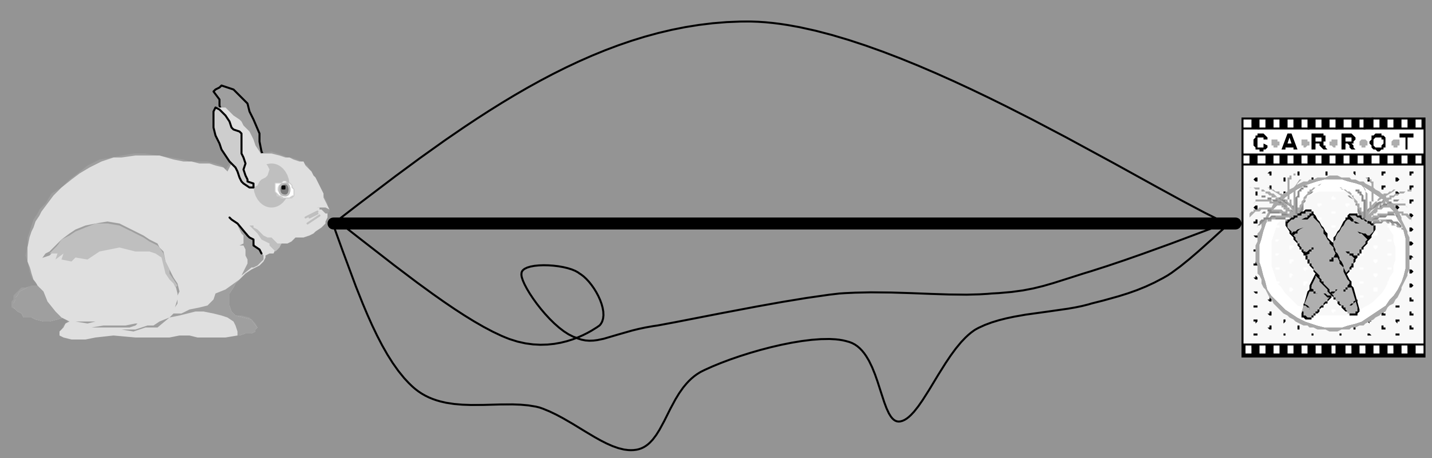
Motivation: What is a Geodesic?

$$d_S^g(p, x) = \inf_C \int_p^x g(C)\,ds$$



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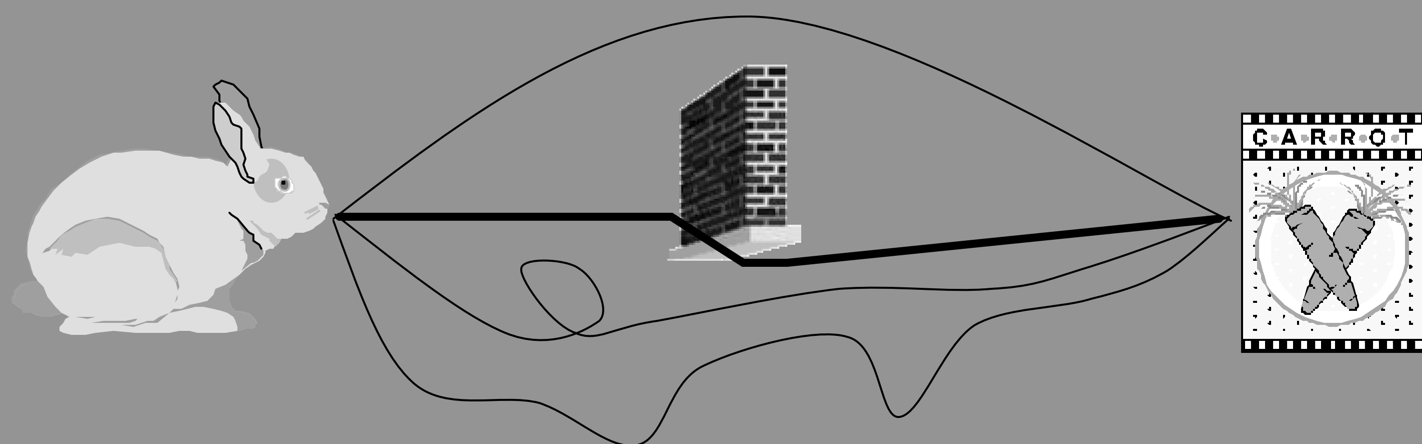
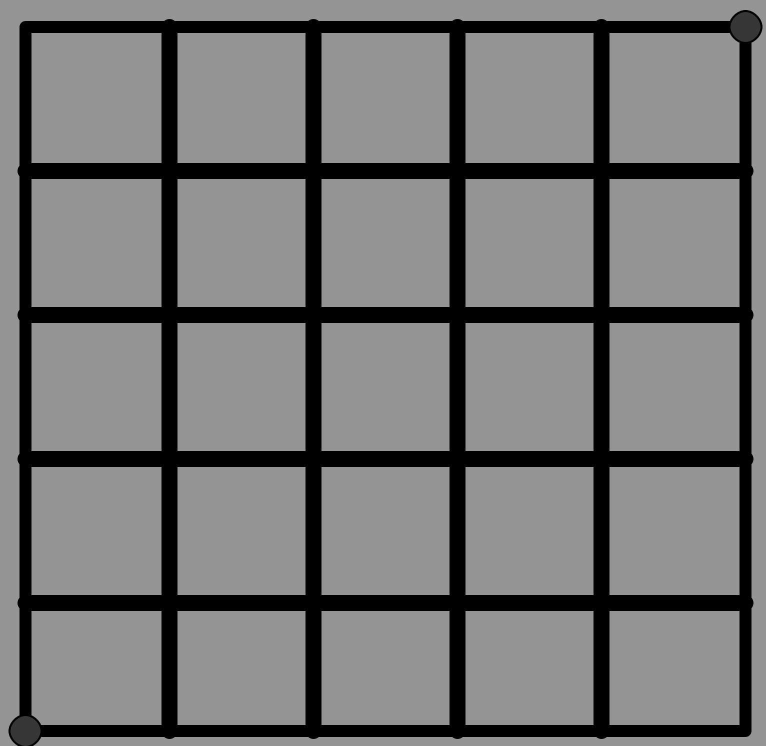


Background: Distance and Geodesic Computation via Dijkstra



- **Complexity: O(n log n)**
- **Advantage: Works in any dimension and with any geometry (graphs)**
- **Problems:**
  - Not consistent
  - Unorganized points?
  - Noise?
  - Implicit surfaces?

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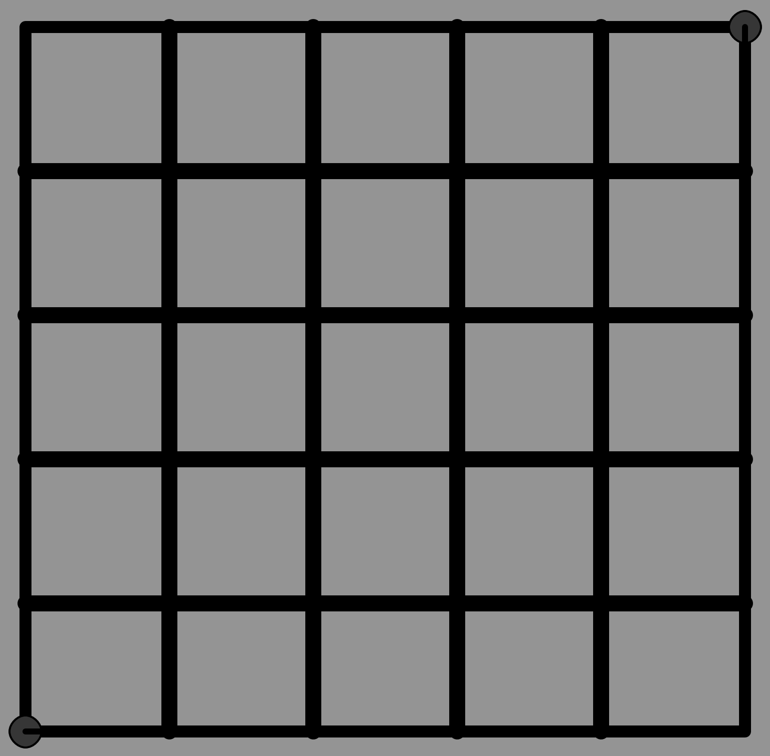

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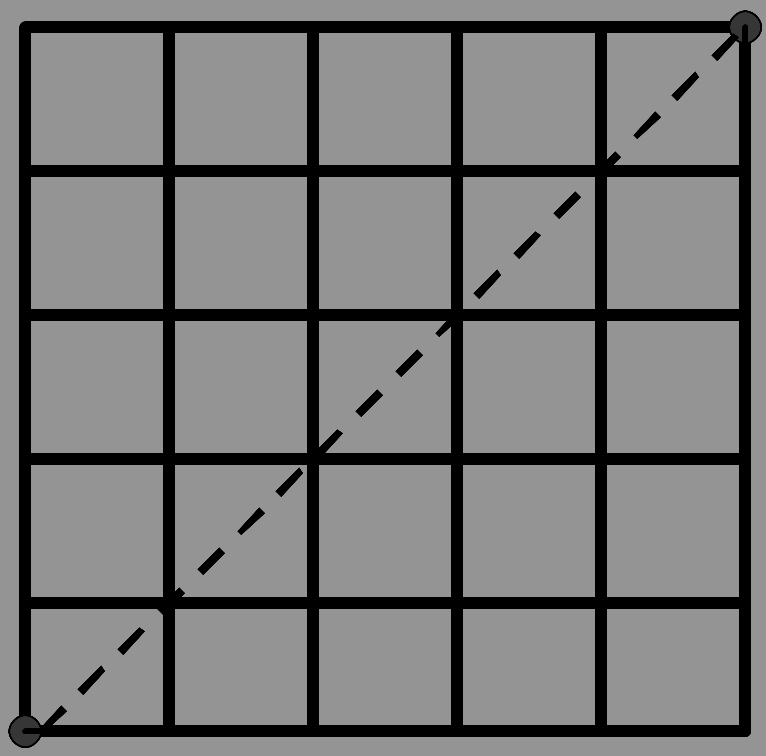

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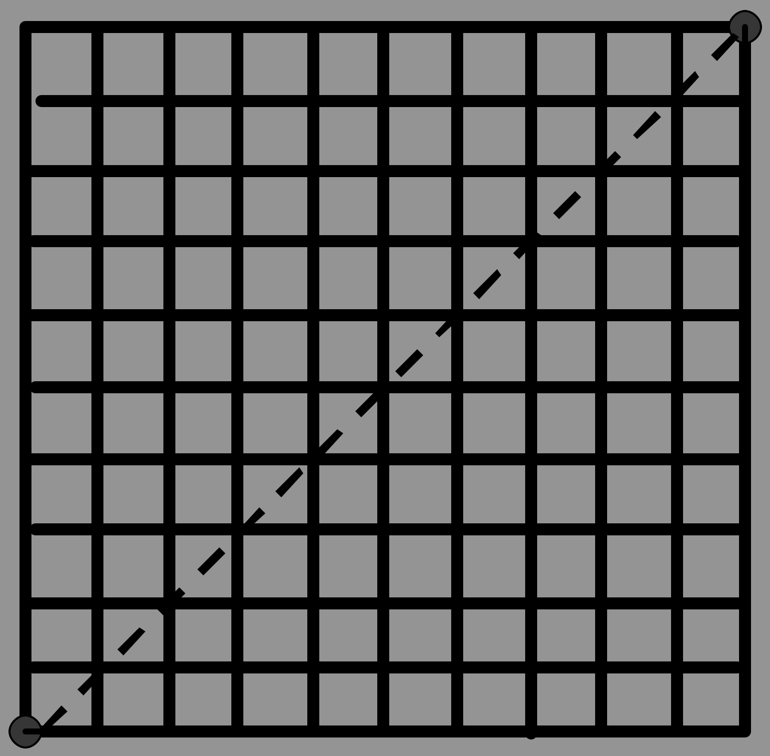

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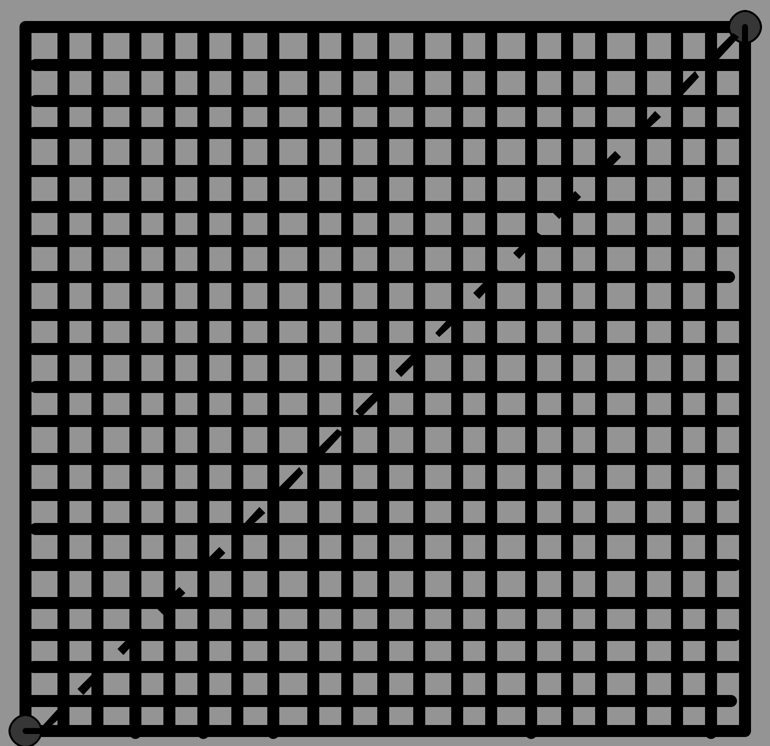

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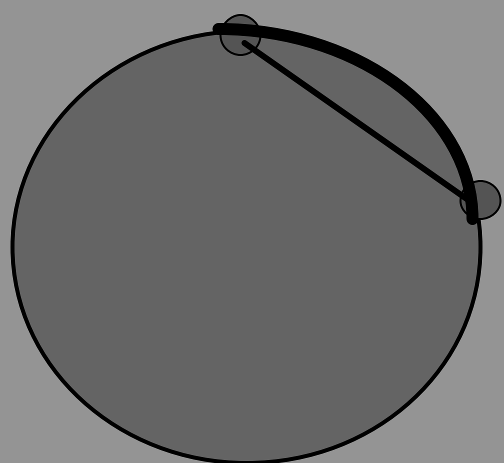
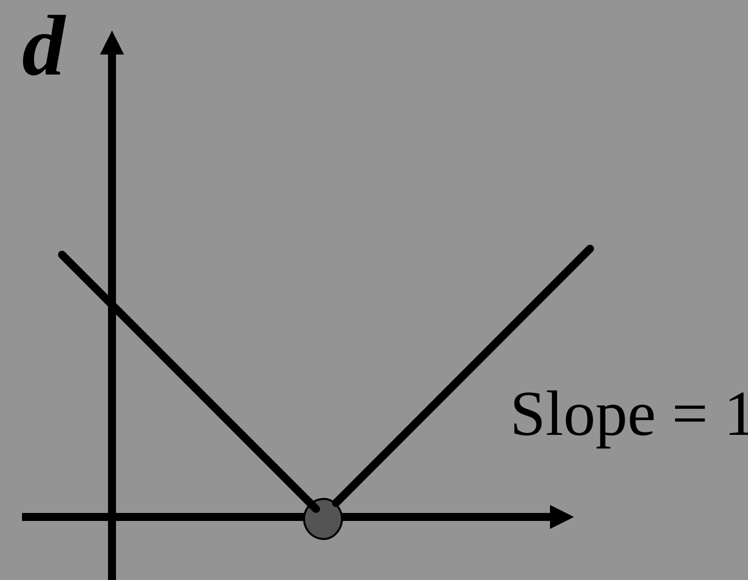
Background: Distance Functions as Hamilton-Jacobi Equations

- ***g*** **= weight on the hyper-surface**
- **The *g*-weighted distance function between two points *p* and *x* on the hyper-surface *S* is:**

$$\left\| \nabla_S d_S^g(p, x) \right\| = g$$

$$\left\| \tilde{N}_S \, d_S^{g}(p, x) \right\| = g$$



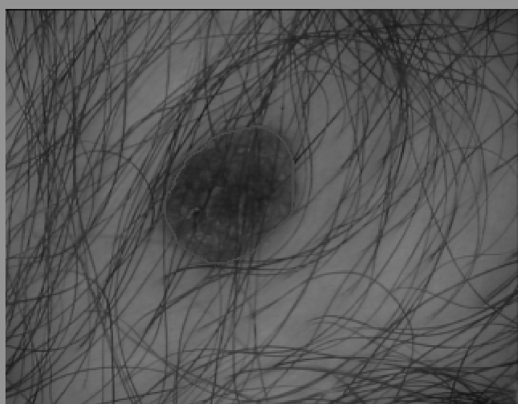


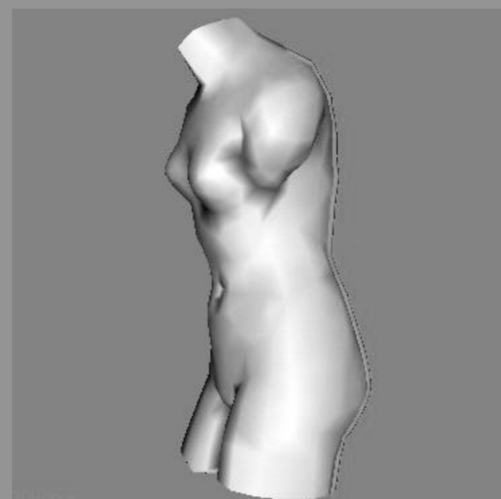
Background: Computing Distance Functions as Hamilton-Jacobi Equations

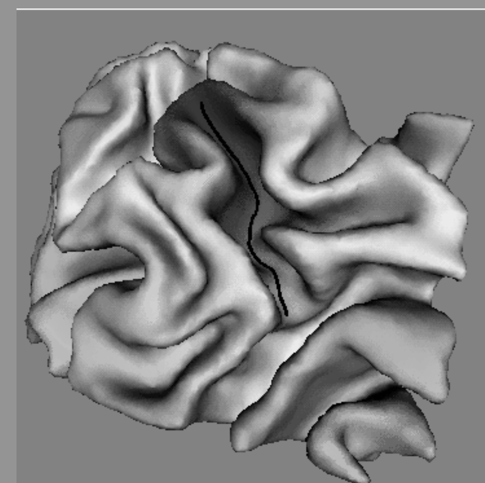
- **Solved in O(n log n) by Tsitsiklis, by Sethian, and by Helmsen, only for Euclidean spaces and Cartesian grids**

$$\left\| \nabla d^{g}(p, x) \right\| = g$$

- **Solved only for acute 3D triangulations by Kimmel and Sethian**



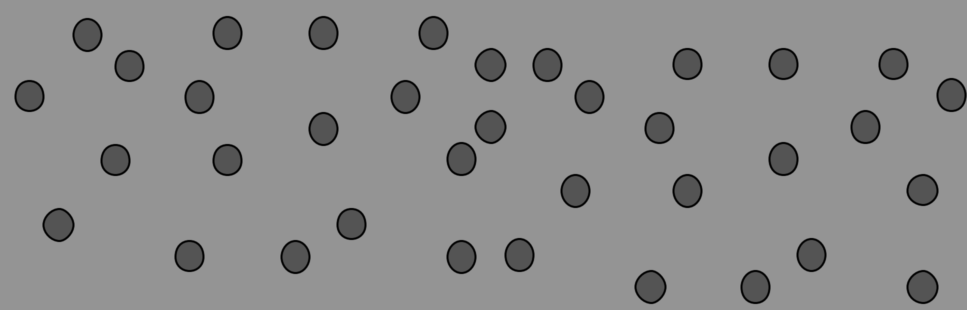




The Problem

- **How to compute intrinsic distances and geodesics for**
  - General dimensions
  - Implicit surfaces
  - Unorganized noisy points (hyper-surfaces just given by examples)

Intermezzo:
Tenenbaum, de Silva, et al...



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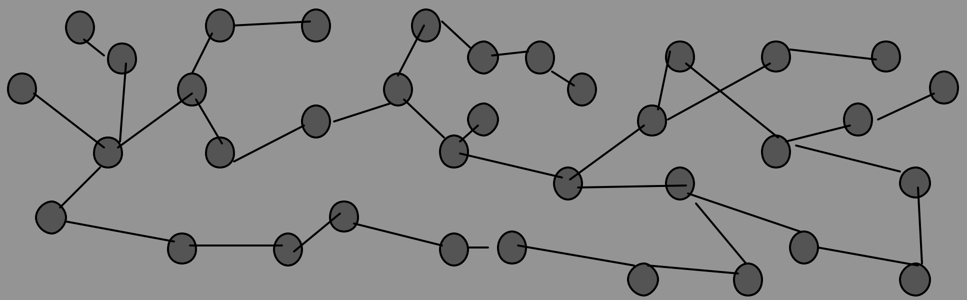

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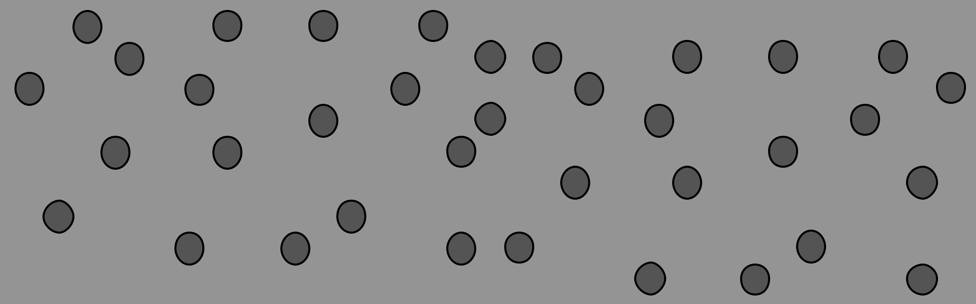


- **Problems:**
  - Doesn’t address noisy examples/measurements
  - Restriction on sampling density and manifolds
  - Uses Dijkestra (back to non consistency)
  - Doesn’t work for implicit surface representations

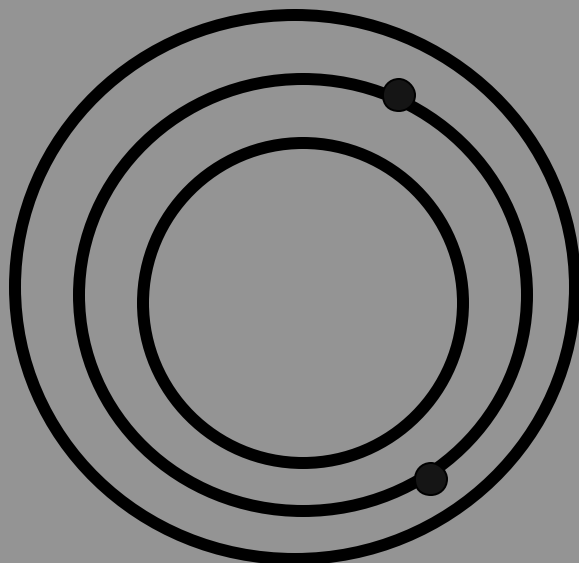
Our Approach

- **We have to solve**

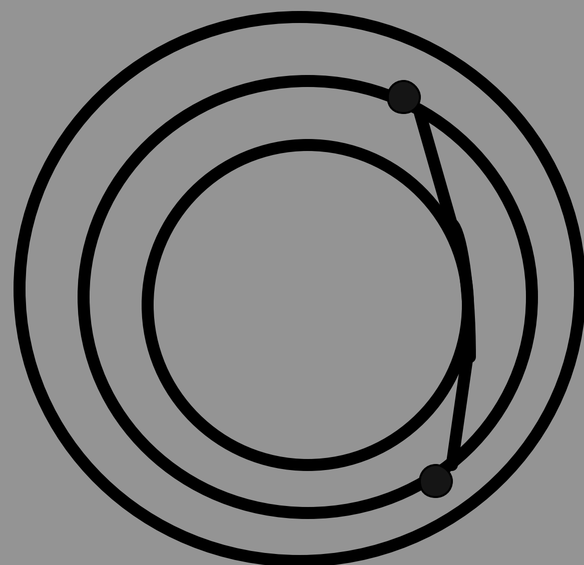
$$\left\| \tilde{N}_{s} \boldsymbol{d}_{s}^{g}\boldsymbol{(p, x)} \right\| = \boldsymbol{g}$$

Basic Idea

Basic Idea



Basic Idea

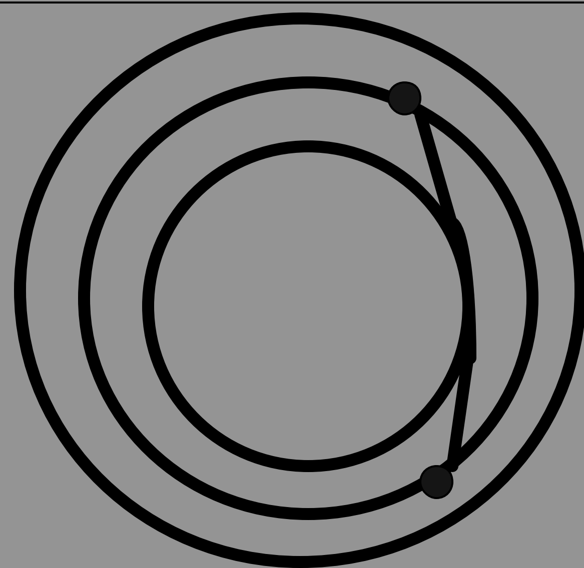


Theorem **(Memoli-Sapiro):**

(open/closed -- any co-dimension)

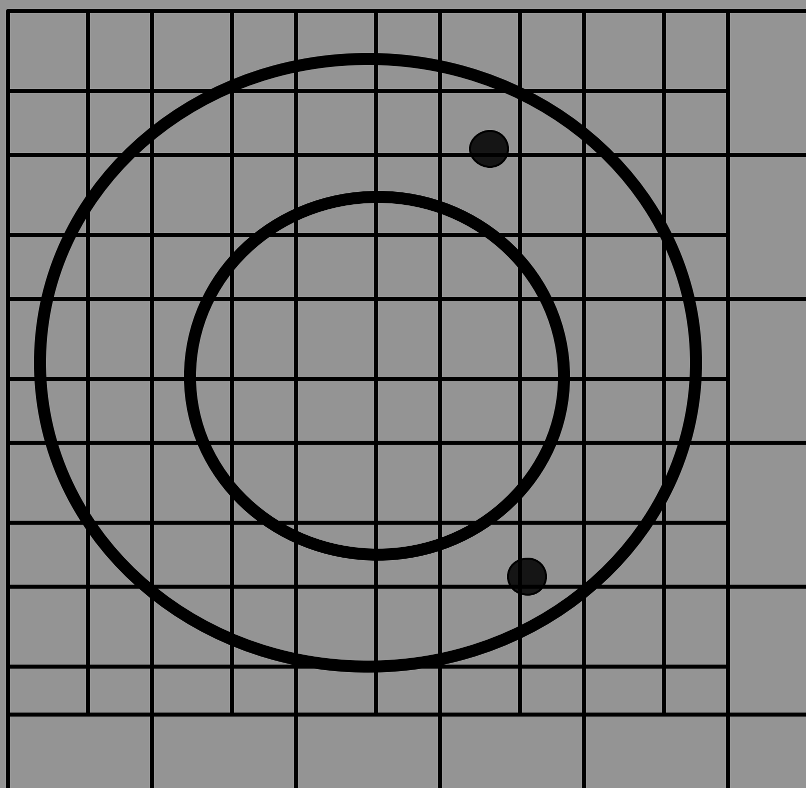
$$\left| d^{g} - d_{S}^{g} \right| \rightarrow 0$$

Basic idea



$$\left| d^{g} - d_{S}^{g} \right| \rightarrow \begin{cases} h^{1/2} & \text{general} \\ h & \text{local analytic} \\ h^{g},\ g > 1 & \text{"smart" metric} \end{cases}$$

Why is this good?



$$\left\| \nabla_S d_S^{\gamma}(p, x) \right\| = \gamma$$

$$\Downarrow$$

$$\left\| \nabla d^{\gamma}(p, x) \right\| = \gamma$$

Implicit Form Representation

$$S = \text{level-set of } \Psi : R^n \to R = \{x : \Psi(x) = 0\}$$

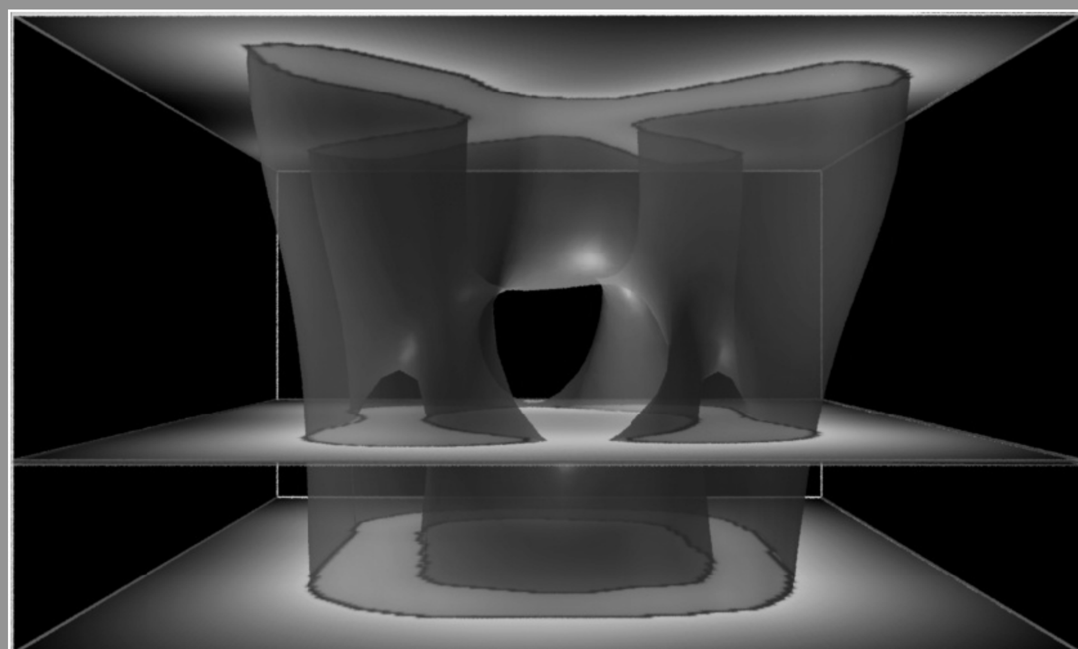


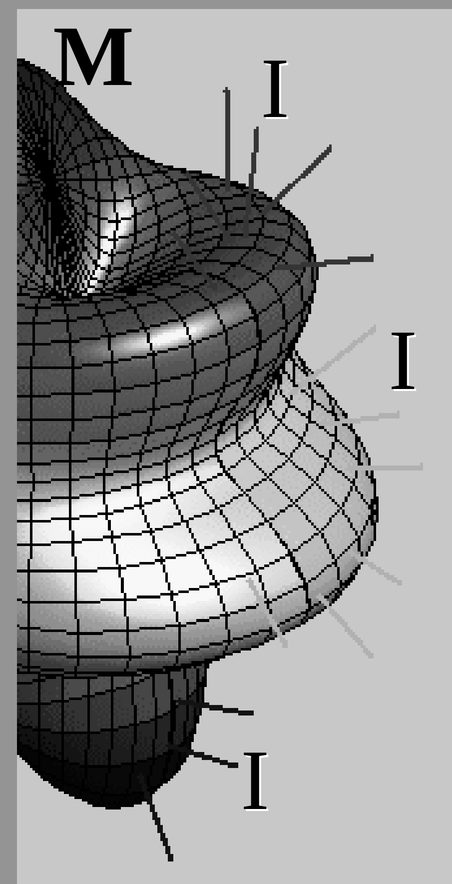
Figure from G. Turk

Data extension

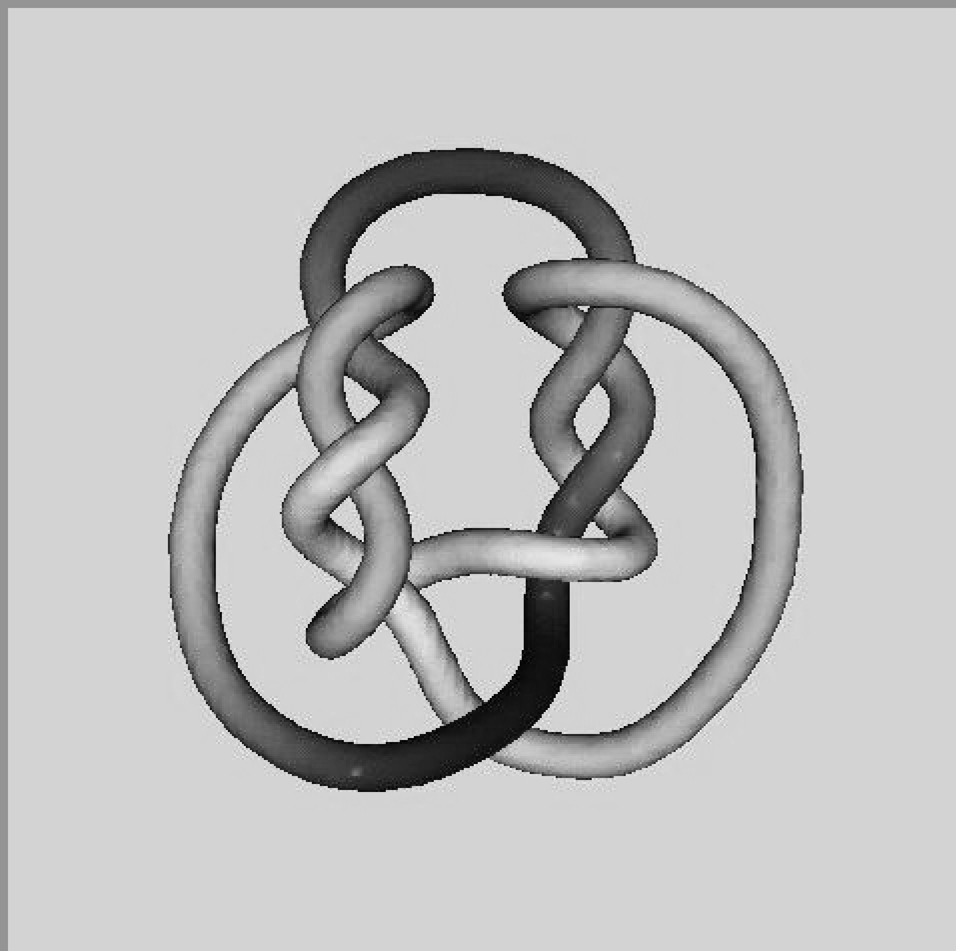
I : M ® R

- **Embed M:**

- **Extend I outside M:**



Examples



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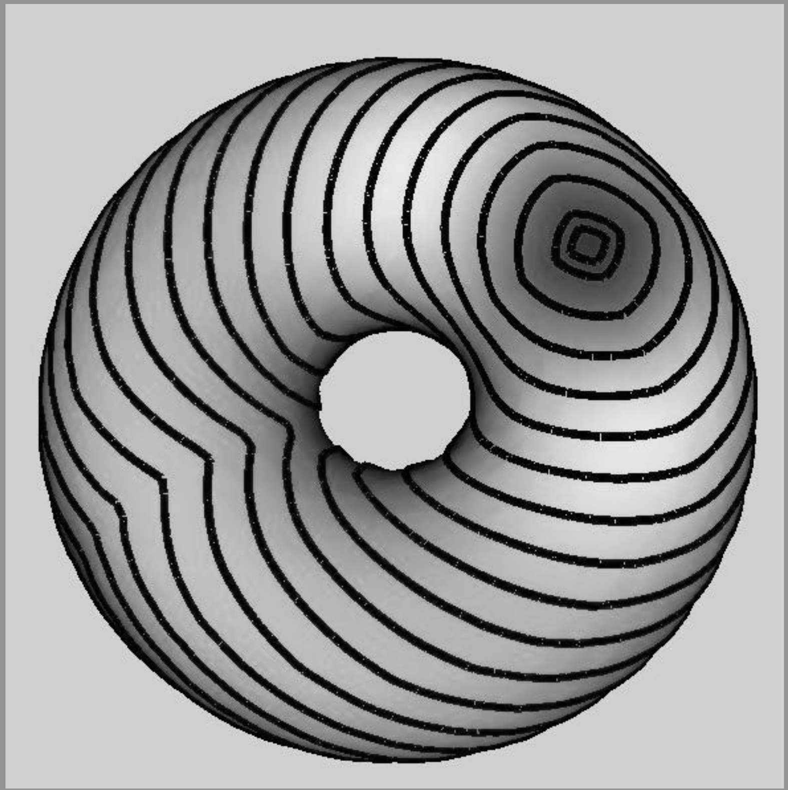

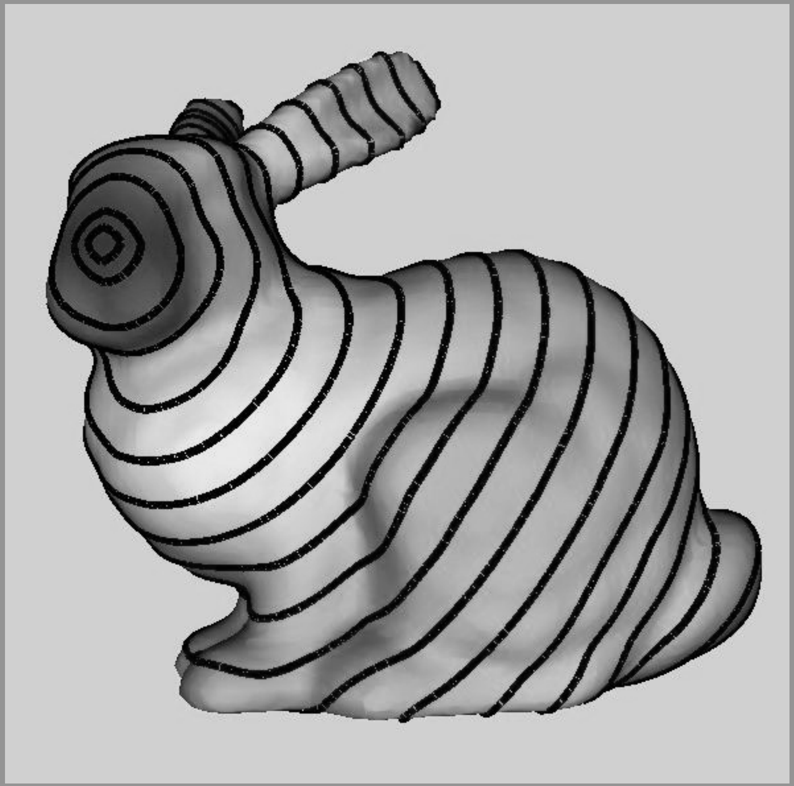

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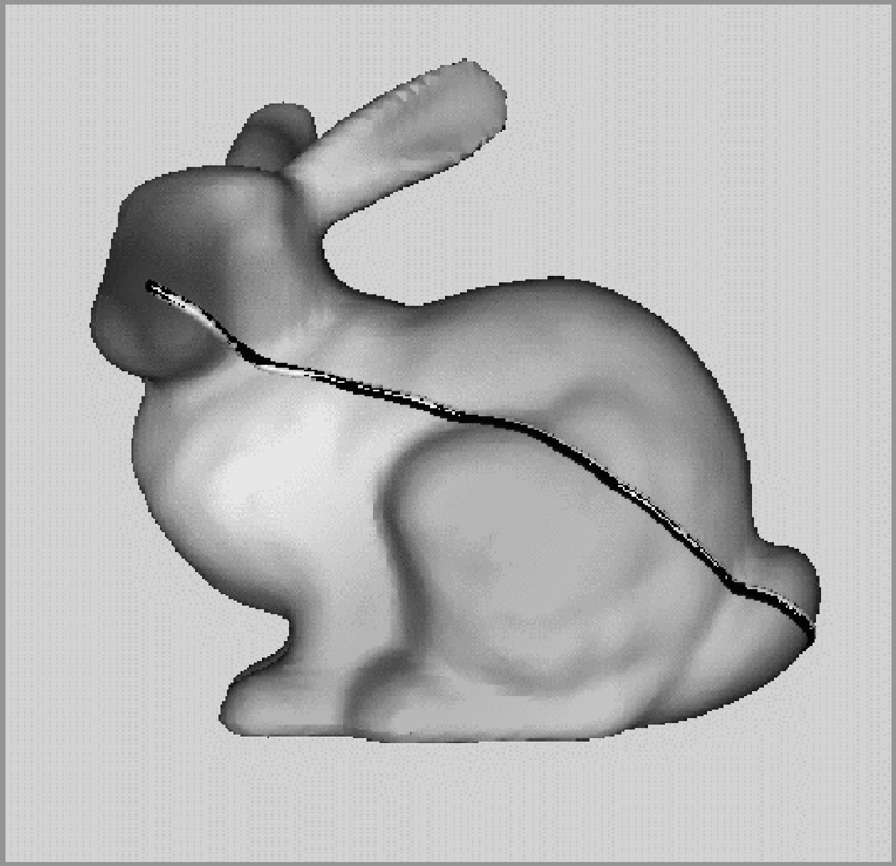

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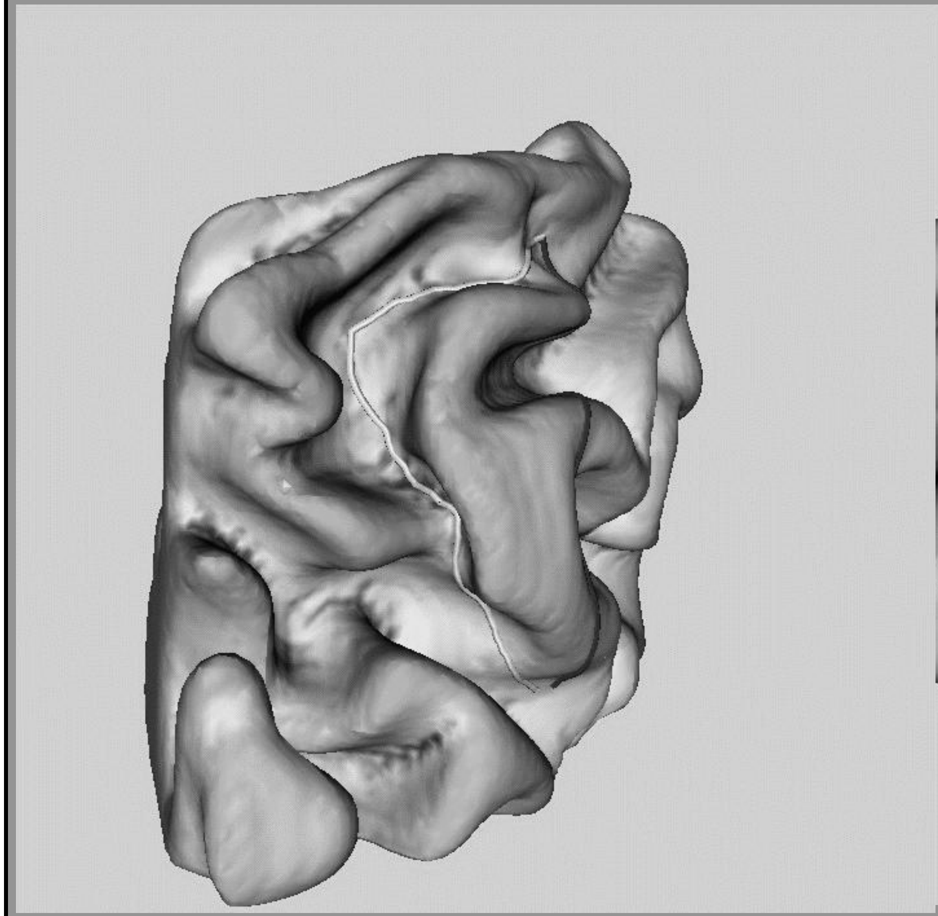

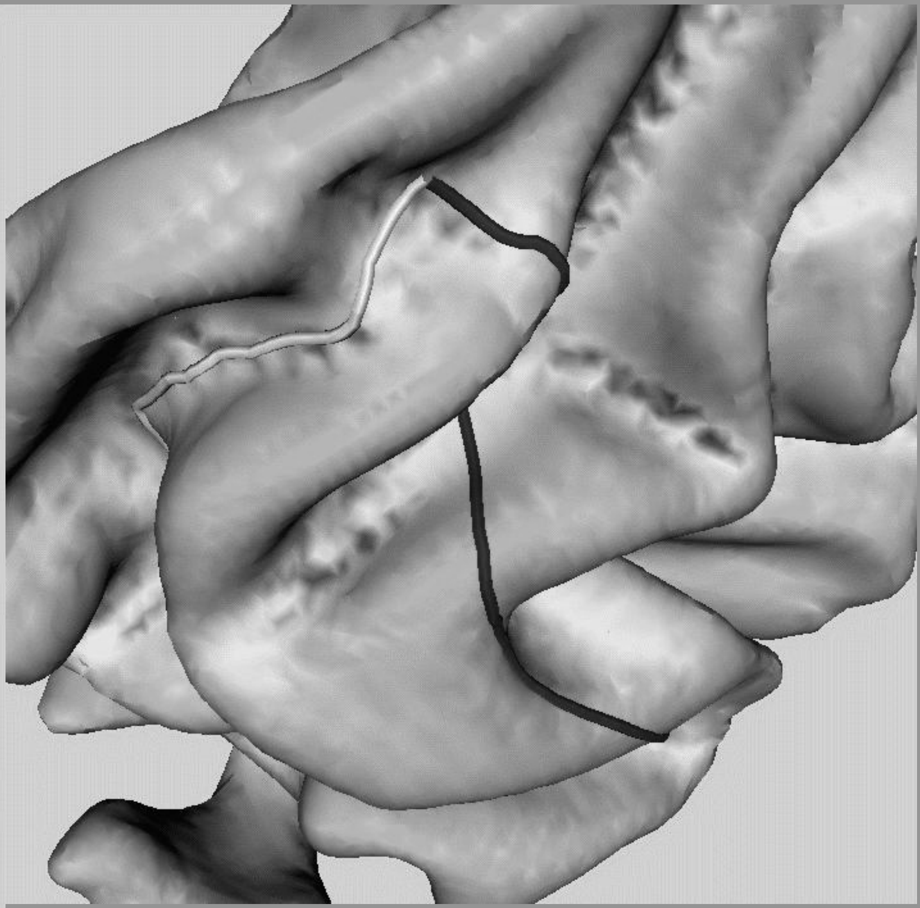

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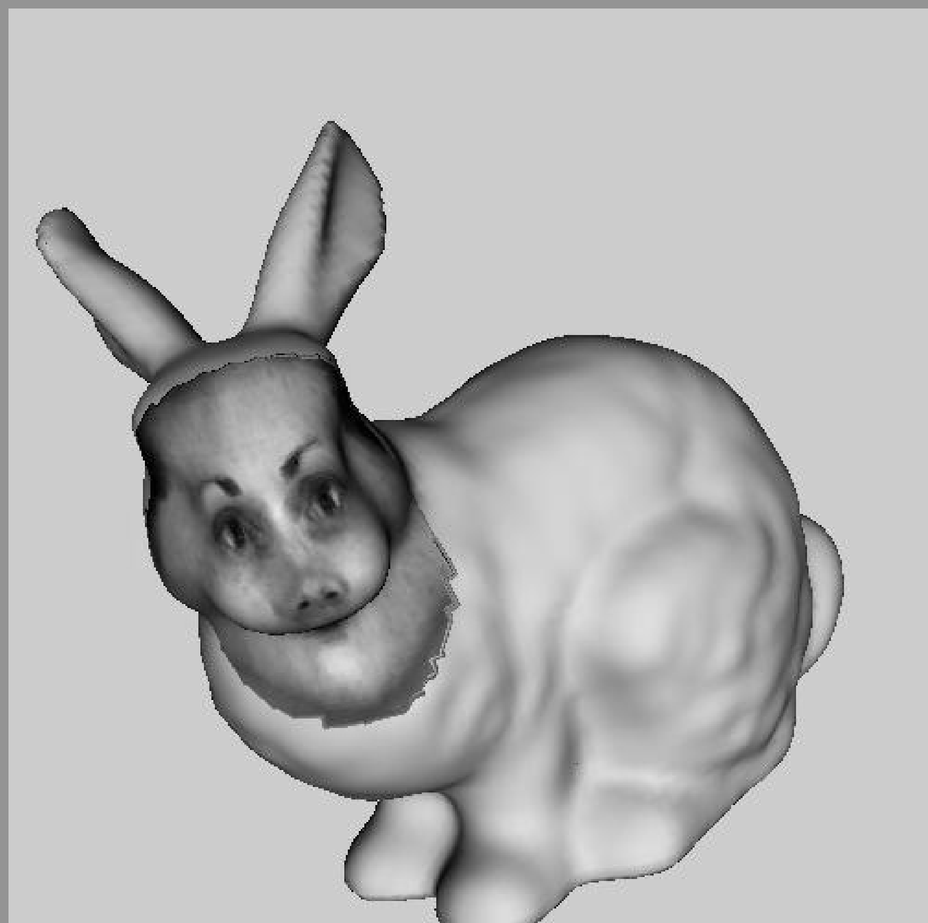
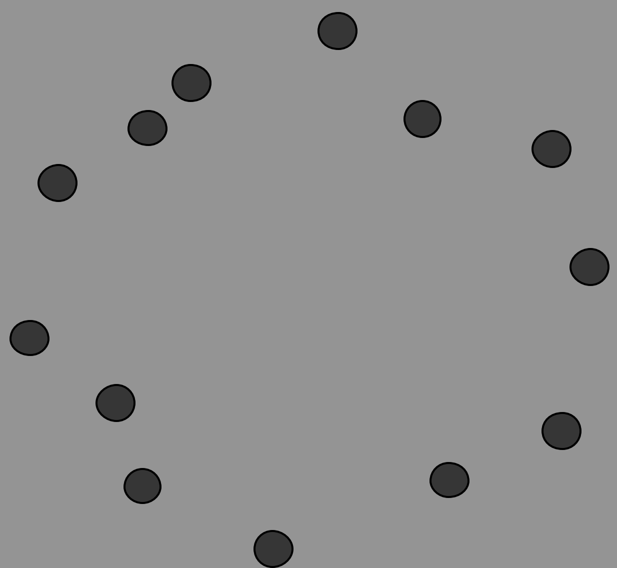
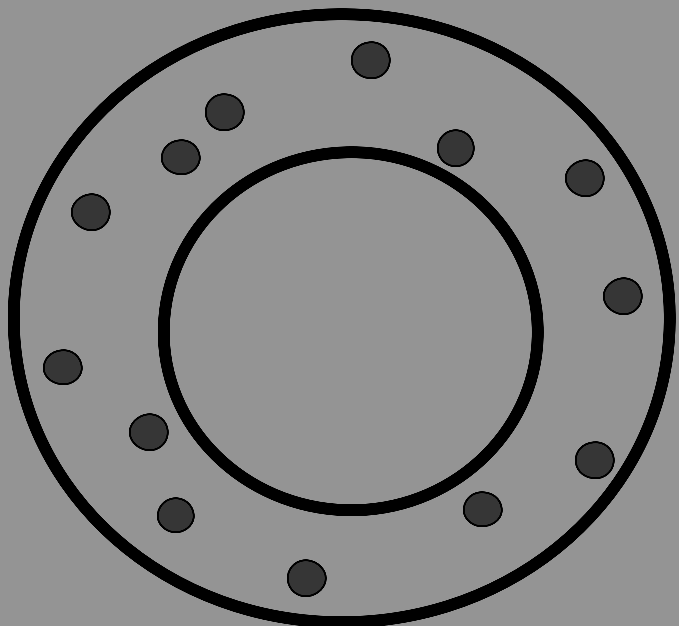


Unorganized points



Unorganized points (cont.)



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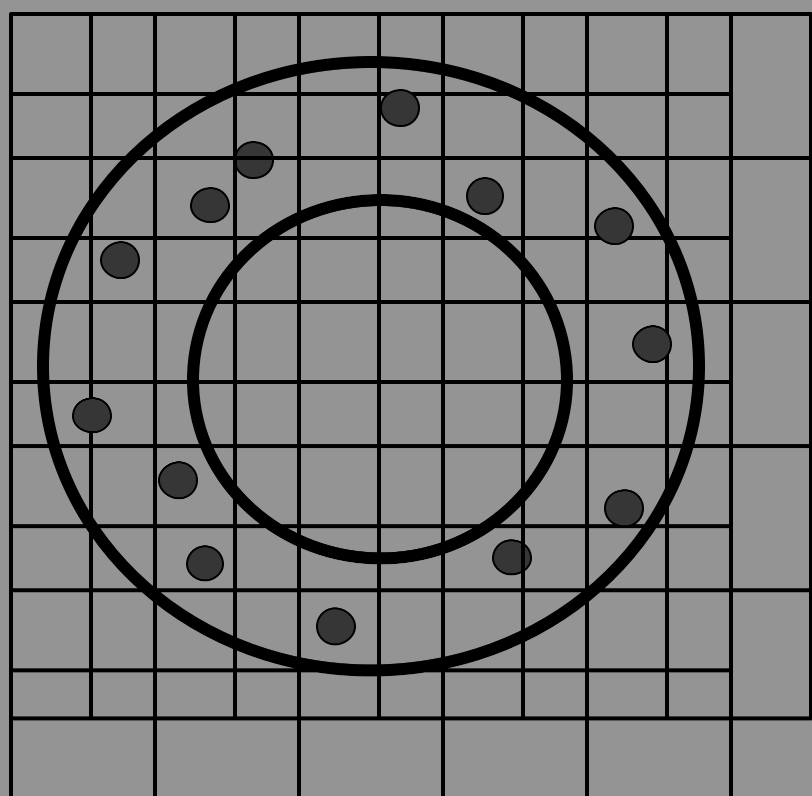
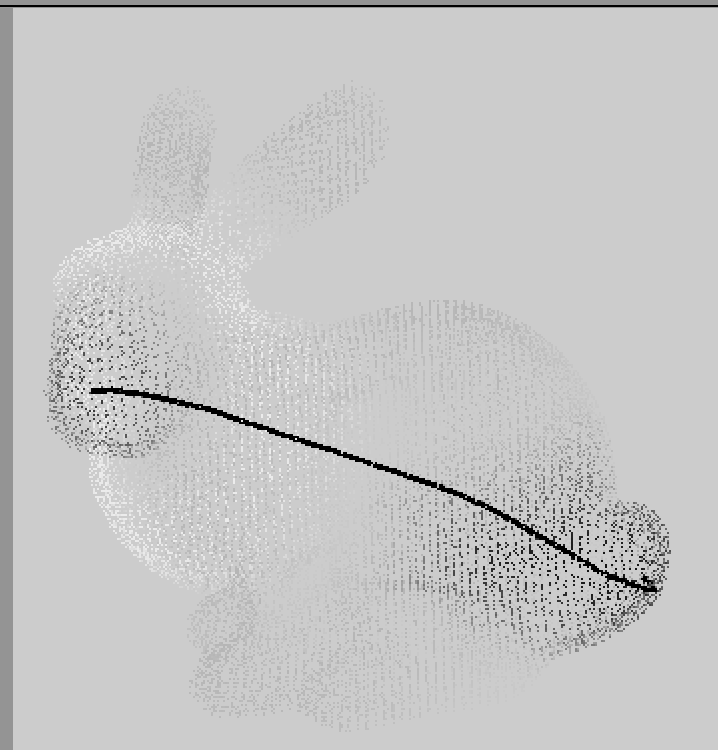
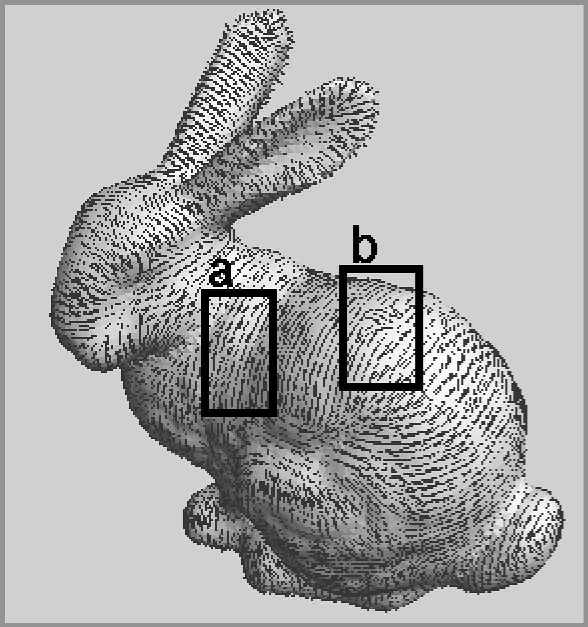


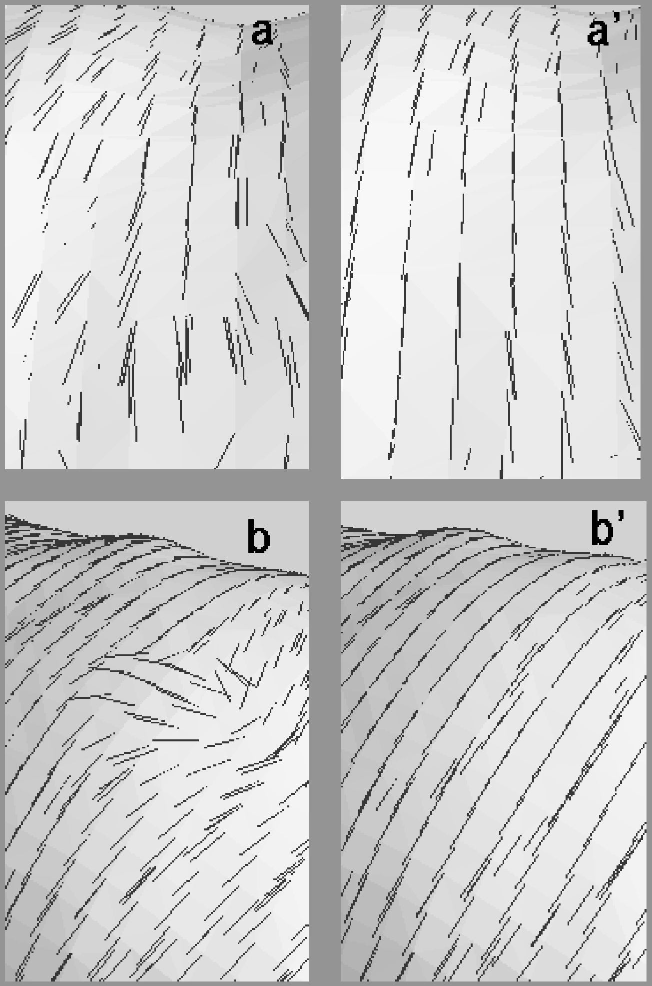


$$\left| d^{g} - d_{S}^{g} \right|_{h,N} \rightarrow 0$$

Is this a geodesic?



noisy



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Generalized geodesics: Harmonic maps

- **Find a smooth map from two manifolds *(M,g)* and *(N,h)* such that**

$$\min_{I:M\to N}\int_{\Omega}\left\|\nabla_M I\right\|^p \, dvol_M$$

$$\left(\frac{\partial I}{\partial t}=\right)\qquad \Delta_M I + A_N(I)<\nabla_M I,\nabla_M I>=0$$

Examples

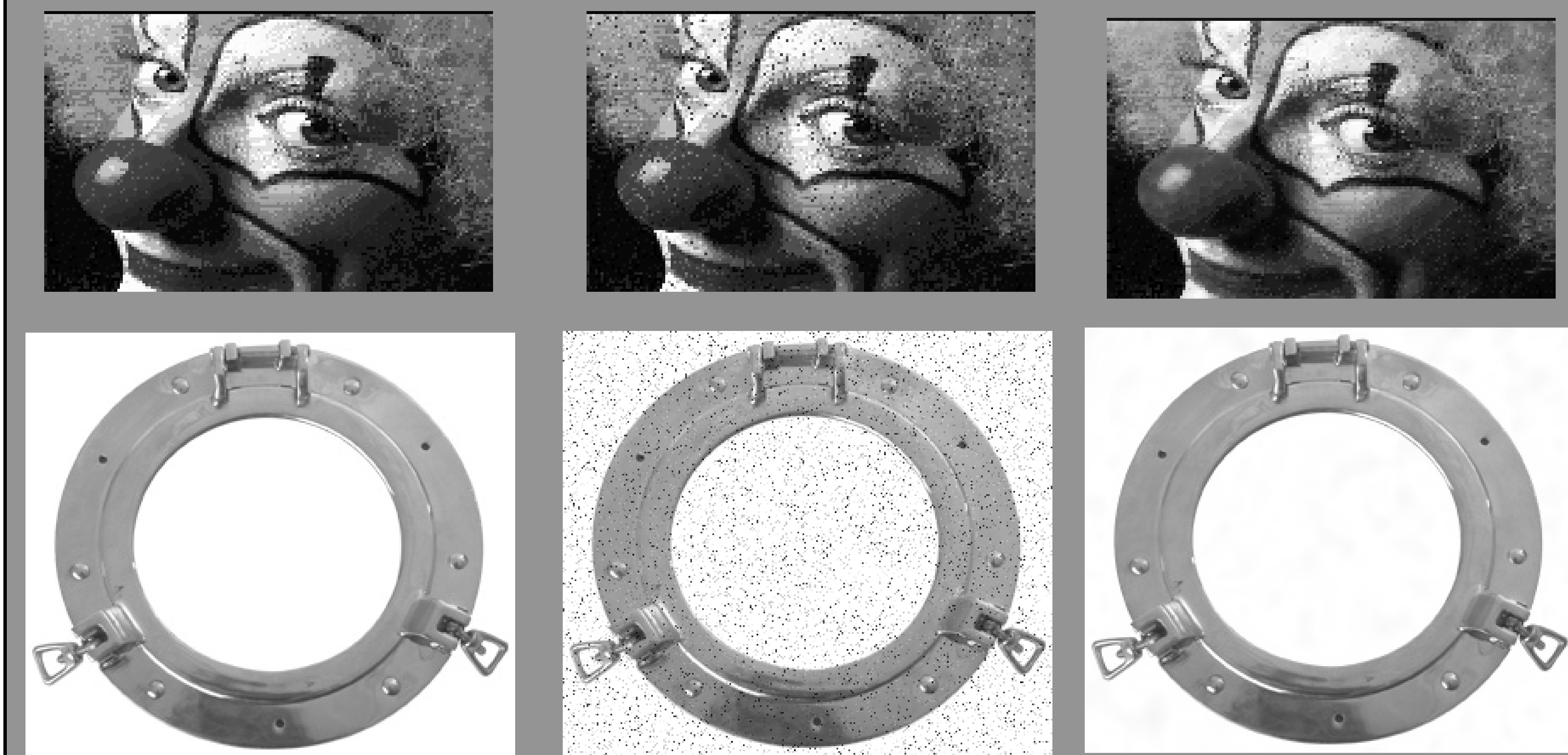
- **M is an Euclidean space and N the real line**

$$\Delta \boldsymbol{I} = \mathbf{0}$$

- **M = [0,1], geodesics!**

$$\frac{\partial^2 \boldsymbol{I}}{\partial \boldsymbol{t}^2} + \boldsymbol{A}_{\boldsymbol{N}}(\boldsymbol{I}) < \nabla_{\boldsymbol{M}} \boldsymbol{I}, \nabla_{\boldsymbol{M}} \boldsymbol{I} >= \mathbf{0}$$

Color Image Enhancement

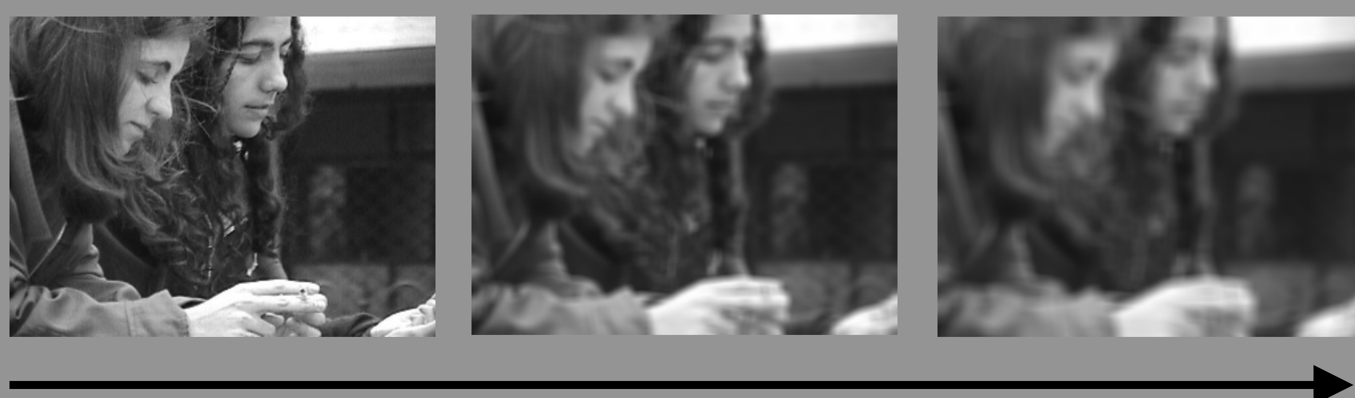


Generalized geodesics: Harmonic maps

- **How we implement this?**
  - Consider M and N defined in implicit form.

Heat flow on the plane

$$\boldsymbol{E} = \int \left\|\nabla \boldsymbol{I}\right\|^2 \Rightarrow \frac{\partial \boldsymbol{I}}{\partial \boldsymbol{t}} = \Delta \boldsymbol{I}$$



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Embedding the domain surface

- **Example: I:M->R**
  - A map from a generic domain surface onto the real line

$$\min_{I:M\rightarrow R} \int_M \left\| \nabla_M I \right\|^2 dvol_M$$

$$\frac{\partial I}{\partial t} = \Delta_M I$$

Embedding the domain surface (cont.)

$$\int_M \left\|\nabla_M I\right\|^2 dvol_M = \int_M \left\|P_{\nabla\Psi}\nabla I\right\|^2 dvol_M$$

$$= \int_{R^3} \left\|P_{\nabla\Psi}\nabla I\right\|^2 \delta(\Psi)\left\|\nabla\Psi\right\| d\mathbf{x}$$

Embedding the domain surface (cont.)

- The gradient descent flow: Heat flow on intrinsic surfaces

- All the computations are done in the Cartesian grid!

Framework

- **If embedding with distance function:**

- **Compare with planar case:**

$$\frac{\partial I}{\partial t} = \operatorname{div}\left(\nabla I\right) = \Delta I$$

Example: intrinsic heat flow

L1 denoising on implicit surfaces

$$\int_M \left\| \nabla_M I \right\| ds = \int_M \left\| P_{\nabla\Psi} \nabla I \right\| \; ds$$

$$= \int_{R^3} \left\| P_{\nabla\Psi} \nabla I \right\| d\,(\bar{\Psi}) \left\| \nabla\Psi \right\| dx$$

$$\frac{\partial \mathrm{I}}{\partial \mathrm{t}} = \frac{1}{\left\| \nabla\Psi \right\|} \mathrm{div}\left(\frac{P_{\nabla\Psi} \nabla I}{\left\| P_{\nabla\Psi} \nabla I \right\|} \left\| \nabla\Psi \right\| \right)$$

L1 denoising on implicit surfaces

$$\frac{\partial \mathrm{I}}{\partial \mathrm{t}} = \mathrm{div}\left(\frac{P_{\nabla\Psi}\nabla I}{\left\| P_{\nabla\Psi}\nabla I \right\|} \right)$$

intrinsic

$$\frac{\partial \mathrm{I}}{\partial \mathrm{t}} = \mathrm{div}\left(\frac{\nabla I}{\left\| \nabla I \right\|} \right)$$

planar

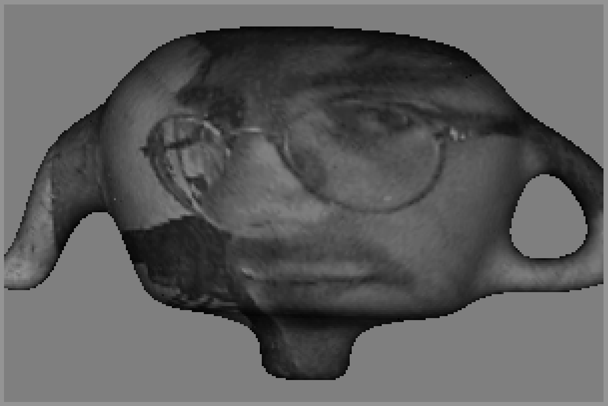
Example:
L1 denoising with constraints

Unit vector/color denoising on implicit surfaces

- **I is a map from the 3D surface to the 3D unit sphere**

$$\frac{\partial I}{\partial t} = \frac{1}{\|\nabla\Psi\|}\,\mathbf{div}\left(\frac{P_{\nabla\Psi}\nabla \boldsymbol{I}}{\|P_{\nabla\Psi}\nabla \boldsymbol{I}\|}\|\nabla\Psi\|\right) + \lambda\,\|P_{\nabla\Psi}\nabla \boldsymbol{I}\|$$

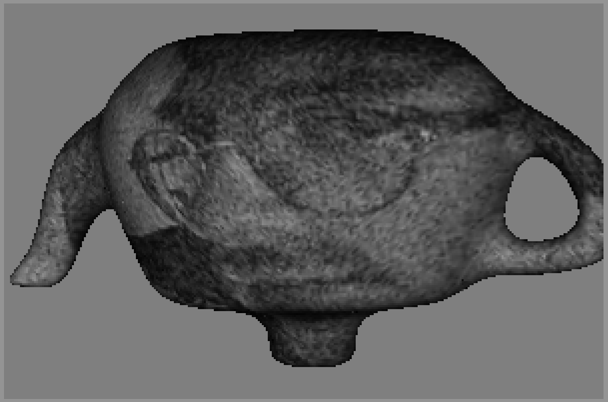
Example: Chroma denoising on a surface

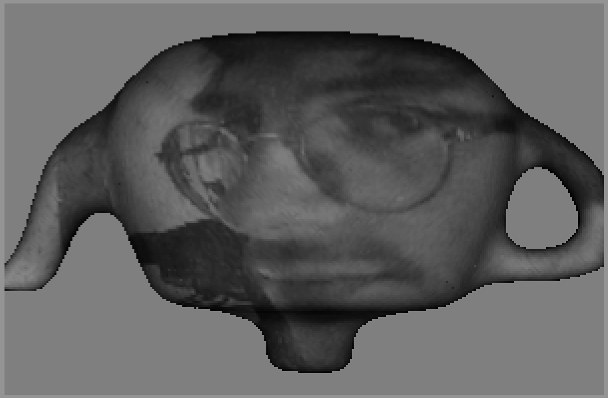


original

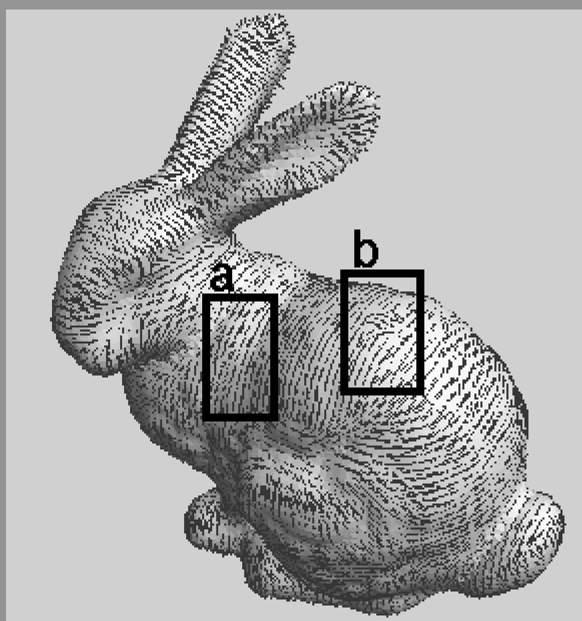
noisy

enhanced





Example: Direction denoising



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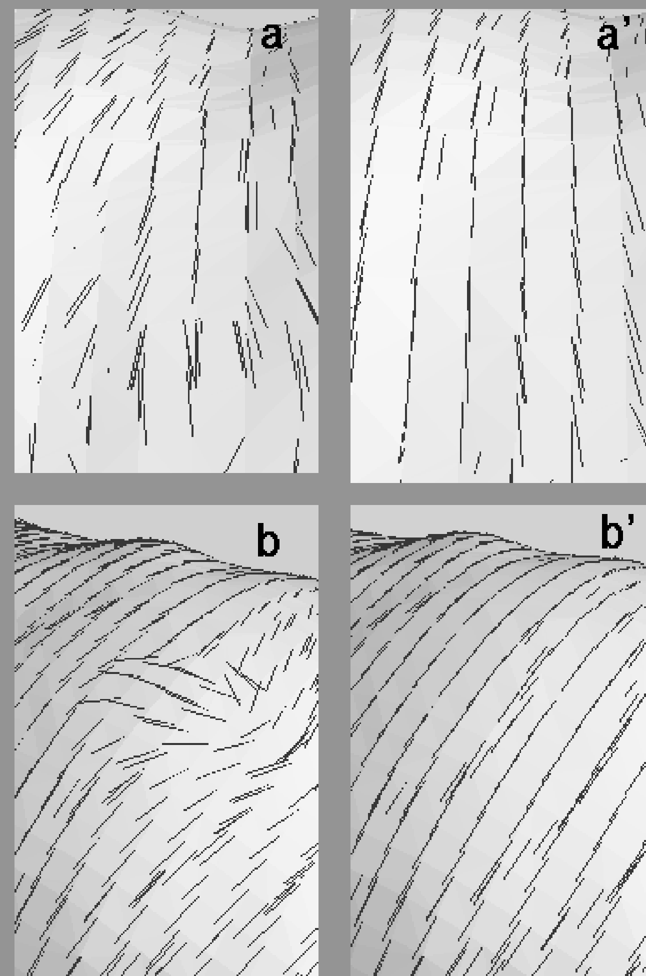

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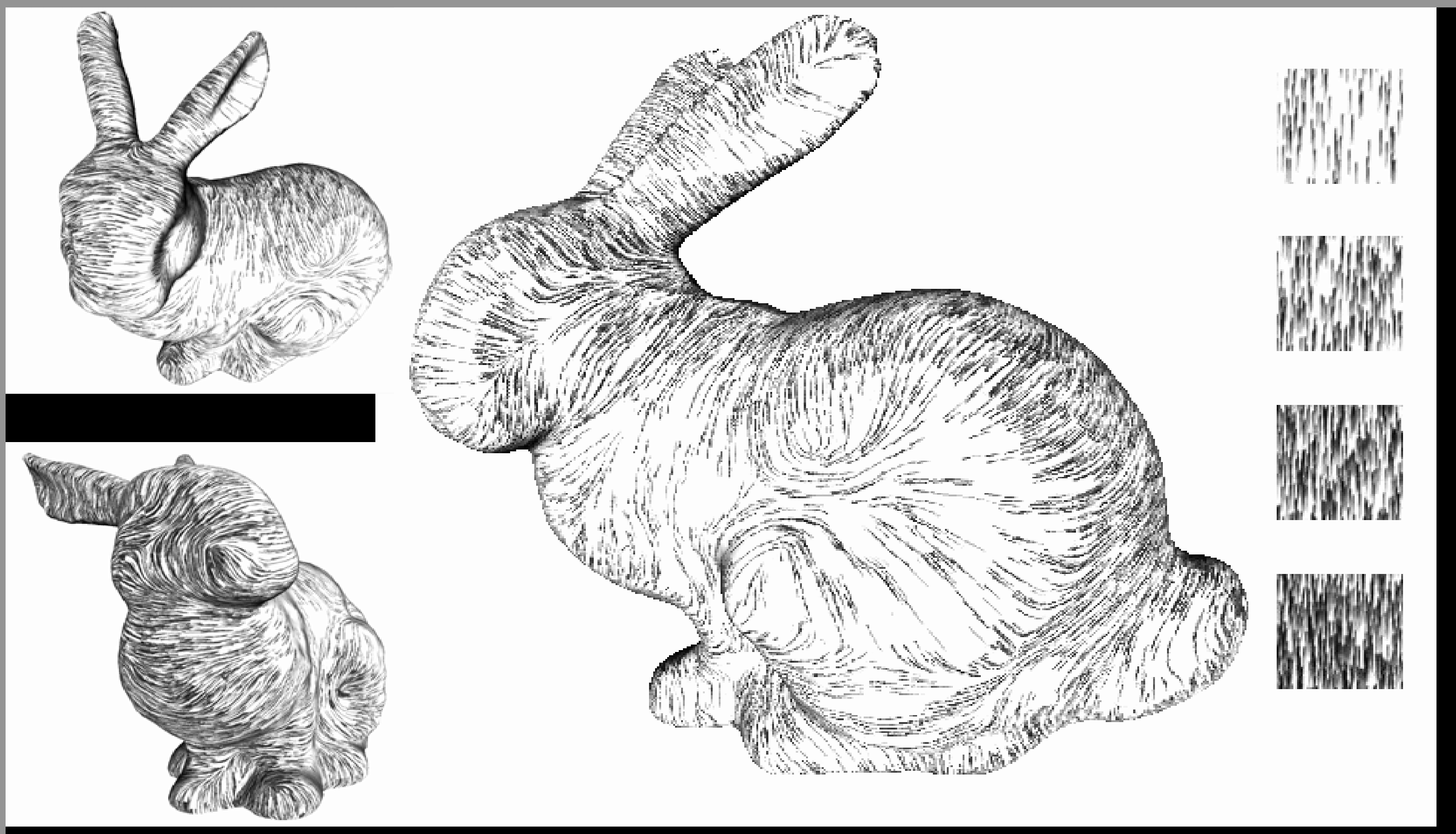
Application (with G. Gorla and V. Interrante)



Pattern formation on implicit 3D surfaces

- **Follows Turing, Kass-Witkin, Turk**

$$
\left.\begin{aligned}
\frac{\partial a}{\partial t} &= f(a,b) + \alpha\, \Delta_M a \\
\frac{\partial b}{\partial t} &= g(a,b) + \beta\, \Delta_M b
\end{aligned}\right\}
\Rightarrow
\left\{\begin{aligned}
\frac{\partial a}{\partial t} &= f(a,b) + \alpha\, \frac{1}{\|\nabla\Psi\|}\,\mathbf{div}\left(\mathbf{P}_{\nabla\Psi}\,\mathbf{I}\,\|\nabla\Psi\|\right) \\
\frac{\partial b}{\partial t} &= g(a,b) + \beta\, \frac{1}{\|\nabla\Psi\|}\,\mathbf{div}\left(\mathbf{P}_{\nabla\Psi}\,\mathbf{I}\,\|\nabla\Psi\|\right)
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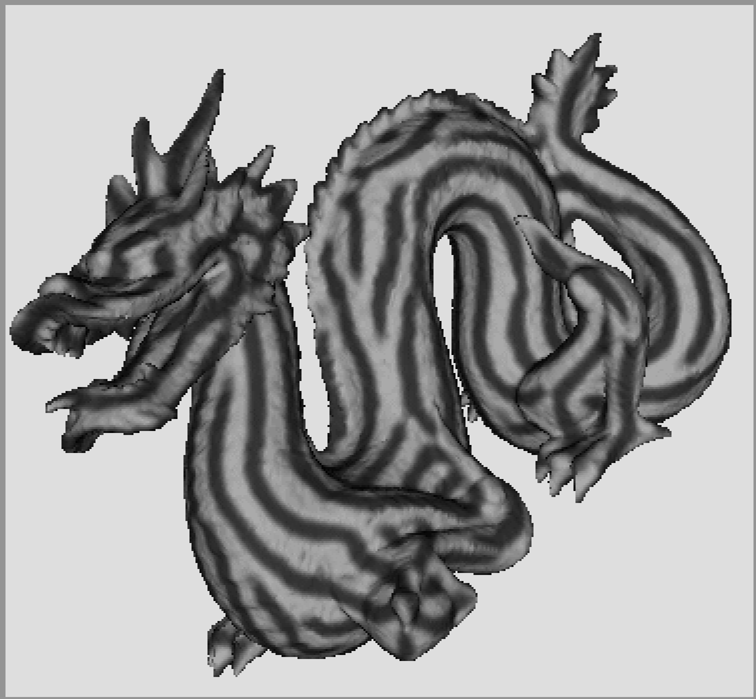
$$

Examples

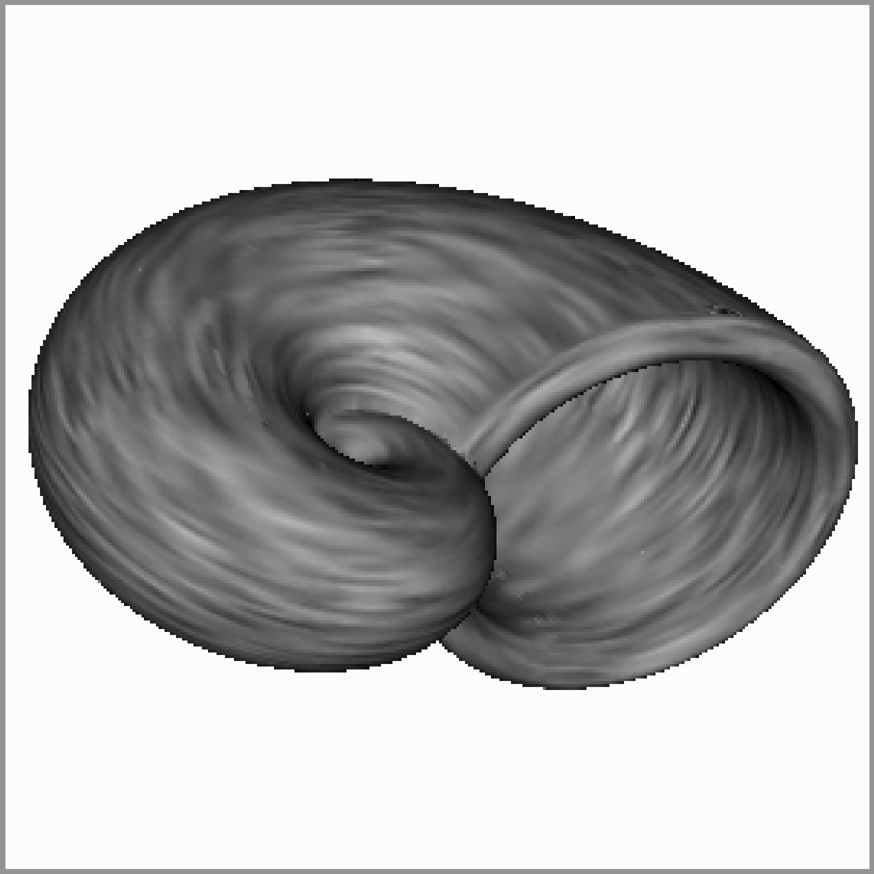


Vector field visualization

- **I is random noise, diffused in direction *v***

$$\left.\begin{array}{l}\dfrac{\partial I}{\partial t} = \operatorname{div}(\mathrm{A}\nabla I) \\ \mathrm{A} = \vec{v}^{T}\vec{v}\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\dfrac{\partial I}{\partial t} = \dfrac{1}{\|\nabla\Psi\|}\operatorname{div}\left(\mathrm{A}\mathrm{P}_{\nabla\Psi}\nabla \mathrm{I}\,\|\nabla\Psi\|\right) \\ \mathrm{A} = \vec{v}^{T}\vec{v}\end{array}\right.$$

Vector field visualization (e.g., principal directions)



Embedding the target manifold

- I:M->N

$$N = \text{level-set of } \Phi = \{x : \Phi(x) = 0\}$$

$$\min_{I:M \to \{\Phi=0\}} \int_M \left\| \nabla_M I \right\|^p dvol_M$$

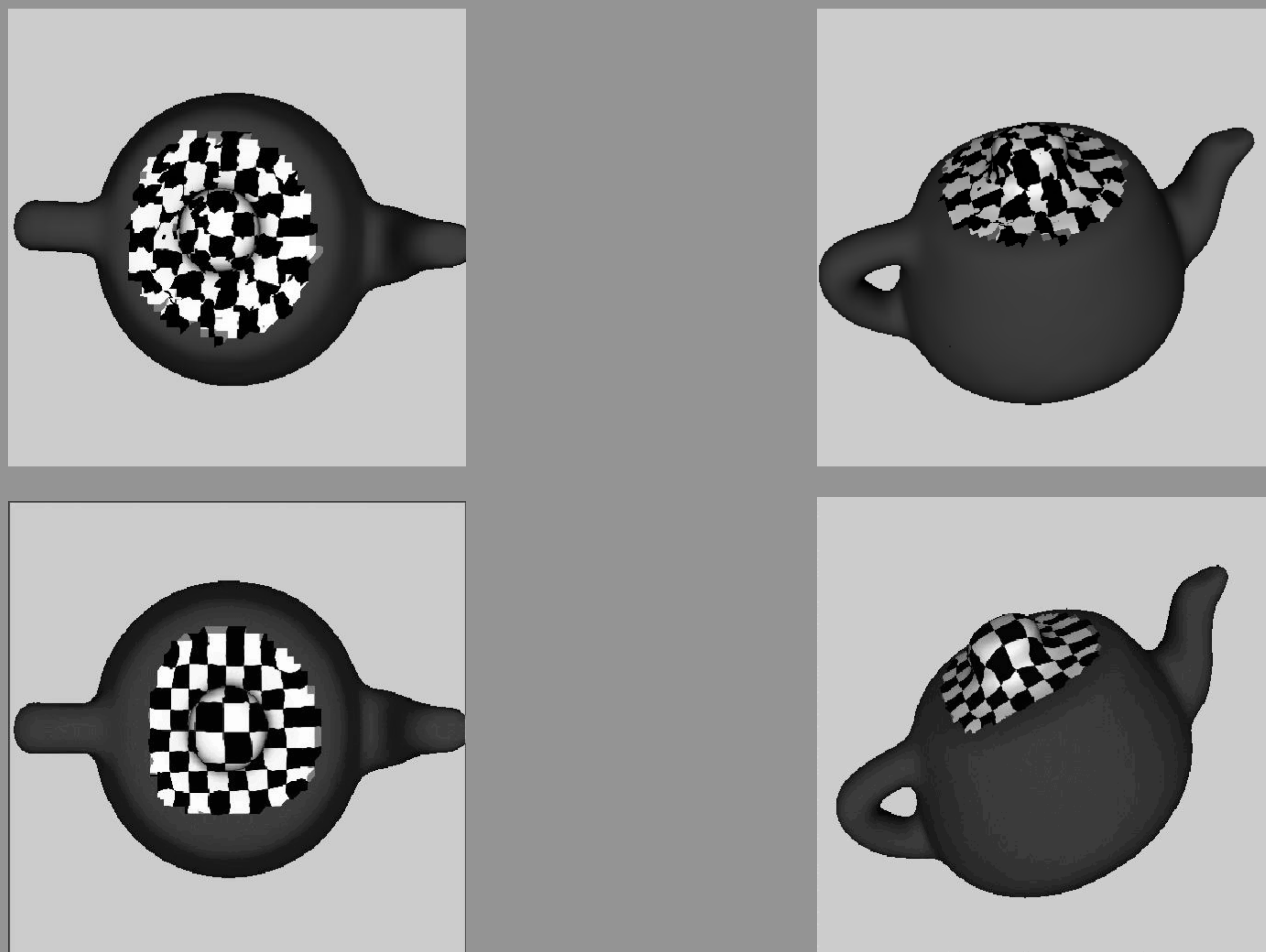
$$\frac{\partial I}{\partial t} = \Delta_M I + A_{\{\Phi=0\}}(I) < \nabla_M I, \nabla_M I >$$

Embedding the target manifold (cont.)

$$\min_{\lambda : R^k \to \{\Phi = 0\}} \int_{R^k} \left\| \nabla_M \lambda \right\|^2 dx$$

$$\frac{\partial \lambda}{\partial t} = ?\,\lambda + \left(\sum_k H_\Phi \left\langle \frac{\partial \lambda}{\partial x_k}, \frac{\partial \lambda}{\partial x_k} \right\rangle \right) \left\| \nabla \Phi \right\|$$

Texture mapping denoising



Texture mapping denoising

Complete Implicit Surfaces Representation

- **Domain and target are implicitly represented: Simple Cartesian numerics**

$$\frac{\partial I}{\partial t} = \mathbf{div}\left(P_{\nabla ?}\, \nabla I\right) + \left(\sum_{k} H_F \left\langle \frac{\partial I}{\partial x_k}, \frac{\partial I}{\partial x_k} \right\rangle \right) \left\| \nabla F \; \right\|$$

- **Extended also to sub-manifolds via intersection of implicit surfaces**

Concluding remarks

- A general computational framework for distance functions, geodesics, and generalized geodesics
- Implicit hyper-surfaces and points clouds

Thanks

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